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BOUNDARY CONDITIONS ON CONTINUUM THEORIES OF GRANULAR FLOW

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Abstract—Continuum theories of highly agitated granular flows have recently been developed based on ideas from the kinetic theory of gases, with the fluctuation velocity of the grains corresponding to the temperature of the gas. Most often the boundary conditions for a granular system at a wall have been taken to be the same as the boundary conditions for a gas (i.e. the "no-slip" boundary conditions on the average flow velocity and the temperature). However, it is clear from experimental observations that a significant slip can exist in the average flow velocity and temperature at a wall.

In this paper, a model of boundary conditions on granular flows will be presented which incorporates the following points:

1. The average flow velocity of the grains at the wall does not equal the wall velocity, with the shear stress at the wall being proportional to the difference in these velocities (the "slip velocity").
2. Small-amplitude vibrations of the wall can be regarded as one factor in an effective wall "temperature". The other factor is the effect of the roughness of the wall coupled with the slip velocity. The flux of "thermal" energy between the granular system and the wall is determined by the relative values of this effective wall "temperature" and the granular system "temperature".
3. Due to differences between grain–grain and grain–wall collisions, the density of the granular system may exhibit a "jump" at the wall.
4. For walls of insufficient roughness, measured angles of effective internal friction may reflect more the effect of shearing at the wall than in the bulk.

These boundary conditions are illustrated by solving a problem in Couette flow.

Key Words: boundary conditions, granular systems, kinetic theory

INTRODUCTION

The assumption of binary collisions in highly agitated granular systems has led to the development of several similar theories for describing such systems (Ogawa *et al.* 1980; Haff 1983; Jenkins & Savage 1983; Lun *et al.* 1984). These theories utilize ideas from the kinetic theory of gases, modified to include the effects of inelastic collisions, to obtain continuum equations for the momentum and energy in the bulk system.

In order to solve practical problems with these theories, boundary conditions are required which relate the parameters of the granular system adjacent to a wall to the forces and velocities associated with that wall. In the continuum equations, these boundary conditions are usually given as:

1. The flow velocity at the wall is set equal to the wall velocity (the no-slip condition).
2. The temperature of the system at the wall is set equal to the wall temperature.
3. The density of the system at the wall is assumed to be unaffected by the presence of the wall.

However, in the case of granular systems, these simple conditions can no longer be used:

1. Experimental evidence indicates that slip between the flow velocity and the wall velocity is a common feature of granular flow problems (Hanes 1987).
2. In the kinetic theory of gases, the wall is often treated as a single, infinitely large, infinitely massive particle (Henderson *et al.* 1976; Waisman *et al.* 1976). If we apply this model to the case of granular flow, then the second moment of the wall's

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velocity may be regarded as the wall's "temperature". This temperature will be determined by the balance between three effects: the generation of thermal energy by the shear stresses and slip velocity; the conduction of thermal energy between the wall and the particles; and the loss of thermal energy due to the inelasticity of grain–wall collisons. Since these mechanisms differ in details from the corresponding mechanisms in the bulk of the flow, the temperature of the wall may be significantly different from the temperature of the particles at the wall.

3. These differences between grain–wall and grain–grain collisions may also cause the bulk density of grains at the wall to differ from the bulk density away from the wall. If, for example, the rebound velocity of a particle in a grain–wall collision were lower on average than the corresponding rebound velocity in a grain–grain collision, then each grain at the wall would occupy a smaller volume of space than a grain in the bulk would occupy. Thus, the bulk density (or equivalently the number density) of grains at the wall would be higher than the density away from the wall.

Hui *et al.* (1984) presented a set of boundary conditions for the phenomenological theory of Haff (1983) based upon the rates of energy and momentum transfer at a wall. Although the slip velocity of the grains at the wall was included in the calculation of the momentum transfer, the thermal energy which would be generated by this slip was not included in the derivation of the energy transfer. Thus, the case in which a significant portion of the internal thermal energy of a granular flow is supplied by slip at the wall cannot be treated within this framework.

Another set of boundary condition equations has been given by Jenkins & Richman (1986), both for a two-dimensional system of smooth circular disks as well as a three-dimensional system of spheres. Employing methods of averaging from the kinetic theory of dense gases, they derive expressions for the rate at which linear momentum and energy are transferred between the granular flow and the wall. Equating these expressions to the corresponding rates in the flow gives the boundary conditions. These authors were the first to emphasize the role of the normal stress boundary condition. However, in the application they discuss, a "slip" in number density at the wall was not allowed for, leading to an over-constrained set of equations, i.e. an additional boundary condition had been introduced, on the pressure, but no additional variables. Thus, the solution of a steady-state Couette flow problem required a unique number of flow disks across the gap. The physical difficulty with this result can be seen in the limit of zero wall velocity, where the Couette flow problem reduces to the problem of particles in a box with no flow. In the absence of allowing for a density "slip" at the wall, there is in general no way to arrive at a steady-state population of particles in the box.

The concept of density slip, discussed in detail below, was introduced by Gutt (1987) in order to remedy this condition, and has subsequently been invoked by other authors as well (Hanes *et al.* 1988).

The theory of boundary conditions for three-dimensional systems presented here is derived in a manner similar to that used by Haff (1983) in obtaining the equations of motion for the bulk flow. In this model each microscopic process of interest, such as momentum transfer in grain–grain collisions, momentum transfer in grain–wall collisions, energy absorption in grain–grain collisions and so forth, is considered explicitly, and the corresponding local expressions for energy and momentum transfer within the bulk and between the bulk and the walls are derived. These expressions are suitably averaged and combined in order to arrive at the desired equations of motion, constitutive relations and boundary conditions. This approach does not start with a particle distribution function, contrary to the tack taken in some applications of kinetic theory, and hence it cannot calculate the precise magnitude of the dimensionless coefficients (q, r, t etc.; see below) which characterize each physical process when those processes are combined together in a balance law. On the other hand, in the present model these factors are not arbitrary but are known to be of order unity. Jackson (1986) has discussed how rigorous kinetic theory gives results which differ only slightly from ours. The advantage of the heuristic approach used here is that specific physical processes are identified clearly from the start at the microscopic level, and that their role in the equations of motion, constitutive relations and boundary conditions remains clear by virtue of the unique tag they carry in the form of a specific dimensionless constant.

These constants are not intended to be used as "fitting parameters", but rather as indicators of the importance and role of specific microprocesses. We also note that, over a very wide range in velocities and densities, there may in fact be slight variations in the values of the "constants" of the model (as is also true in kinetic theory). For our purpose here, which is to outline some new and interesting boundary effects in granular systems, we neglect any such variation, which is expected to be small.

Working within this framework, our approach will be to introduce "slips" in the bulk density and thermal velocity, the necessity for which is argued below, as well as the more conventional slip in average velocity, and then to solve the problem of Couette flow, in which the properties of the grains and wall along with the wall velocity and Couette-cell width are given, and the pressure, shear stress and velocity profiles in the bulk are calculated.

It should be noted that in the kinetic theory of granular flow presented by Haff (1983), the bulk density ρ is assumed to be essentially constant throughout the flow. This assumption is adopted here, so that the results apply mainly to dense systems. When small variations in the bulk density would have a significant effect (as in the collision rate), the variations are allowed by the use of the grain-to-grain spacing variable s. We will continue to use this formalism in describing the variation in density at a wall.

THE GRAIN–WALL COLLISION MODEL

The grains are assumed to be identical, inelastic, smooth spheres of diameter d and mass m. (In order to simplify our treatment of the boundary conditions, the spin of the grains will be ignored). The packing fraction is taken to be high (i.e. $s < d$) and the system is taken to be sufficiently agitated that the particles undergo only binary collisions so that the theory of Haff (1983) can be applied. In a collision, the particle is assumed to contact a section of the wall which has a local unit normal vector $\mathbf{k}$ (figure 1) and a coefficient of restitution e_w.

In calculating the results of a grain–wall collision, the mass of the wall will be taken as infinitely greater than the mass of a grain.

The following assumptions are made about wall roughness:

1. At the microscopic level (smaller than a grain diameter), the wall is smooth and frictionless.
2. On a scale slightly larger than a grain diameter, the surface of the wall has a shape consisting of smooth undulations of amplitude less than a grain diameter. The assumption of a small amplitude is based on the idea that any feature of greater amplitude will tend to trap one or more particles, thus "healing" itself and forming a new boundary with small-amplitude undulations. [Surface "healing"

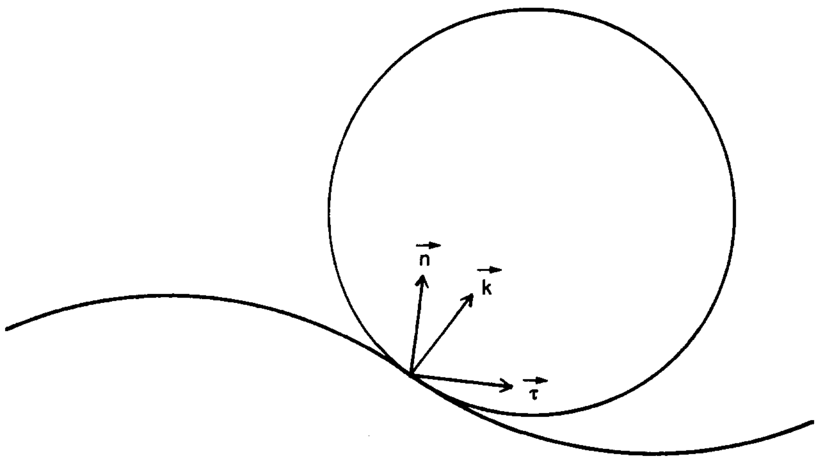

Figure 1. Illustration of a particle colliding with a rough wall, showing the unit normal collision vector $\mathbf{k}$, and the unit vectors normal to the wall ($\mathbf{n}$) and parallel to the wall ($\boldsymbol{\tau}$).

is commonly seen in computer simulations of flow (Haff 1987); see also the discussion of self-bounding fluids in Hui *et al.* (1984).] The surface roughness is characterized by a distribution function for **k**, $f(\mathbf{k})$, which is assumed to be isotropic.

3. On average over distances much larger than a grain diameter, the wall is flat.

In the present treatment of the boundary conditions, the position of the wall as a function of time will be allowed to have a random component. This random component will have an amplitude less than a grain diameter and can vary on a time scale similar to the time between collisions of the grains in the flow. (Motion with larger amplitudes or on longer time scales should be included in the macroscopic description of the boundary's position.) This effective random motion can arise from two uncorrelated sources: the vibrational motion of the wall and the slip velocity.

The vibrational motion of the wall will be described by the second moment of its velocity, designated v_w. Even though this motion will be correlated over the entire length of the wall, it will be treated as contributing an uncorrelated random motion to the grains in view of the fact that the positions of the grains are not correlated over distances greater than a few grain diameters. Further discussion of the analogy between a small-amplitude high-frequency wall vibration and a thermal source is given by Haff (1983).

The random motion due to the slip velocity can be quantified by considering the frame of reference in which the average velocity of the granular flow near the wall is zero. In this frame, the slip velocity coupled with the wall's roughness results in a fluctuation in the normal component of velocity in a grain–wall collision. This component of the surface velocity in the **k** direction (figure 1) is $\mathbf{u}_s \cdot \mathbf{k}$, where $\mathbf{u}_s$ is the slip velocity ($\mathbf{u}_s$ is taken to point along the wall, the $\boldsymbol{\tau}$ direction in figure 1). Averaging this dot product over all possible values of **k**, and adding it in quadrature to any average externally driven vibrational velocity, gives the wall's effective vibrational velocity:

$$v_{w,\,\mathrm{eff}}^2 = v_w^2 + \int (\mathbf{u}_s \cdot \mathbf{k})^2 f(\mathbf{k})\,d\mathbf{k} = v_w^2 + u_s^2 \int (\mathbf{k} \cdot \boldsymbol{\tau})^2 f(\mathbf{k})\,d\mathbf{k} \equiv v_w^2 + u_s^2 \langle k_\tau^2 \rangle. \qquad [1]$$

In this equation, we have explicitly quantified the roughness of the wall in the term $\langle k_\tau^2 \rangle$. A perfectly flat wall will have $\langle k_\tau^2 \rangle = 0$; and an increasingly rougher wall will have increasing values of $\langle k_\tau^2 \rangle$.

Finally, the average rate at which grain–wall collisions take place is given by the slip velocity of the grain and the wall divided by the average grain–wall spacing:

$$\frac{(v^2 + v_w^2 + u_s^2 \langle k_\tau^2 \rangle)^{1/2}}{s_w}, \qquad [2]$$

where v is the average thermal velocity of the grains.

Since these sources of vibrational motion are regarded as random and uncorrelated, they are added in quadrature and the time average of any cross terms is assumed to vanish.

THE BOUNDARY CONDITION EQUATIONS

The pressure

The pressure on the wall can be found by means of the cell model (Hirschfelder *et al.* 1964). The normal component (along **n** in figure 1) of the momentum transferred to the wall in a single grain–wall collision is of order

$$m(v^2 + v_w^2 + u_s^2 \langle k_\tau^2 \rangle)^{1/2}.$$

Since the particle occupies a cell whose dimensions are of order d, the area across which this momentum transfer takes place is approximately d^2. Combining these values with the rate at which grain–wall collisions take place, [2], gives the pressure on the wall:

$$p = t_w d\rho \frac{(v^2 + v_w^2 + u_s^2 \langle k_\tau^2 \rangle)}{s_w}, \qquad [3]$$

where all of the proportionality constants have been incorporated into t_w, a dimensionless constant of order 1.

The pressure in the granular flow (as given by Haff) is $p = td\rho v^2/s$, where t is a dimensionless constant of order 1. Setting these two expressions equal gives the value of the grain–wall spacing:

$$s_w = \frac{t_w}{t}\left(\frac{v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle}{v^2}\right)s. \qquad [4]$$

This boundary condition is the result of the fact that a particle in the layer adjacent to the wall "sees" a different environment on one side of its cell from the others. In order to transmit a constant pressure in the direction perpendicular to the wall, the grain–wall spacing must adjust accordingly. This is equivalent to a "slip" or jump in the bulk density of the system at the wall. [Although density and mean free path do not stand in a strictly one-to-one relation at high density, because of geometrical packing effects, we equate for the purposes of this paper, "density slip" and "mean free path slip", see Haff (1983).]

This "slip" in the bulk density is actually a first-order approximation to the more complex oscillations in bulk density seen in calculations and simulations of hard-sphere fluids bounded by a flat wall (Henderson *et al.* 1976; Snook & Henderson 1978; Waisman *et al.* 1976). These variations in bulk density can arise even though the particle–particle and particle–wall collisions are perfectly elastic, simply due to the layering effect of the particles near a flat wall (Snook & Henderson 1978). Since the wall we use here is not perfectly flat, it would not be appropriate to go beyond this first-order approximation.

The shear stress

On average, a collision between a grain and a rough wall will have a component of momentum transfer along the direction of the slip velocity. This momentum transfer results in the transmission of shear stress between the wall and the granular flow.

To calculate this shear stress, note that the normal velocity in a grain–wall collision due to the slip velocity is $(\mathbf{u}_s \cdot \mathbf{k})\mathbf{k}$; and its component parallel to the wall is $(\mathbf{u}_s \cdot \mathbf{k})\,(\mathbf{k} \cdot \boldsymbol{\tau}) = u_s k_\tau^2$. Multiplying this by the particle mass and averaging over all possible values of $\mathbf{k}$ gives the average component of momentum transfer parallel to the wall:

$$mu_s \int k_\tau^2 f(\mathbf{k})\,d\mathbf{k} \equiv mu_s\langle k_\tau^2\rangle.$$

Taking into account the area and rate of collisions, the flux of lateral momentum will be

$$\sigma = q_w d\rho u_s\langle k_\tau^2\rangle \frac{(v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle)^{1/2}}{s_w}, \qquad [5]$$

where q_w is a dimensionless constant of order 1.

Equation [5] seems to make a puzzling prediction, namely, that the shear stress vanishes if the slip velocity $u_s = 0$. Yet we know that, for fluids in general, a no-slip boundary condition does *not* imply a vanishing stress. In the microscopic granular flow model of Haff (1983) used here, u_s is a dependent variable, a quantity whose value must be computed. It is not something we can adjust by hand. Therefore, the no-slip condition is not to be specified *a priori* by setting $u_s = 0$, but is a condition which might or might not turn out to have validity in the course of the calculation. And, in particular, the no-slip condition does not mean $u_s = 0$. It only means that u_s is small compared with the total shear U (we set $v_w = 0$ for simplicity). In fact, u_s cannot vanish under any flow conditions, save the no-flow case, as can be seen by noting that all velocities are scaled by U.

The shear stress in the granular flow (Haff 1983) is

$$\sigma = \eta\frac{\partial u}{\partial y} = qd^2\rho\frac{v}{s}\frac{\partial u}{\partial y},$$

where q is a dimensionless constant of order 1.

Equating these two fluxes of lateral momentum gives the boundary condition relating the slip velocity and the normal derivative of the flow velocity:

$$\frac{\partial u}{\partial y} = \frac{q_w}{q}\frac{u_s}{d}\langle k_\tau^2\rangle \frac{(v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle)^{1/2}}{v}\frac{s}{s_w}.$$

Substituting from [4] for the ratio of spacings gives

$$\frac{\partial u}{\partial y} = \frac{q_w}{q}\frac{t}{t_w}\frac{u_s}{d}\langle k_\tau^2\rangle \frac{v}{(v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle)^{1/2}}. \qquad [6]$$

This nonlinear relation between the slip velocity and the shear rate reduces to that obtained by Hui *et al.* (1984) in the limit where the particle fluctuation velocity is much greater than the wall's effective fluctuation velocity. Again, for the same reason as in [5], the flow velocity gradient remains nonzero even when the no-slip approximation is a good one.

The condition of isotropy in the distribution function $f(\mathbf{k})$ allows us to assume collinearity of the shear stress and slip velocity at the wall. If $f(\mathbf{k})$ were anisotropic (e.g. a "washboard" wall surface oriented obliquely to the flow), the shear stress would be related to the slip velocity through a second rank tensor.

The thermal energy flux

The thermal energy of particles colliding with a wall is affected by two competing processes. The inelasticity of grain–wall collisions results in a loss of thermal energy, while the effective fluctuation velocity of the wall will supply thermal energy to the grains.

The loss of thermal energy in a collision due to the particle's thermal velocity and inelasticity will be of order

$$m(1 - e_w^2)v^2.$$

The gain of thermal energy in a collision due to the wall's effective temperature will be of order

$$m(1 + e_w)^2 v_{w,\mathrm{eff}}^2.$$

(Note that by using the wall's effective temperature in this expression we have included the thermal energy generated by the combination of shear stress and slip velocity.) Combining these two effects, substituting for the wall's effective temperature, [1], and factoring in the area and rate of collisions, we get an expression for the flux of thermal energy at the wall:

$$Q = -r_w d\rho\left[v_w^2 + u_s^2\langle k_\tau^2\rangle - \frac{(1 - e_w)}{(1 + e_w)}v^2\right]\frac{(v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle)^{1/2}}{s_w}. \qquad [7]$$

The flux normal to the wall of thermal energy in the granular flow (Haff 1983) is

$$Q = -K\frac{\partial}{\partial y}\left(\frac{\rho v^2}{2}\right) = -rd^2\frac{v}{s}\rho\frac{\partial}{\partial y}\left(\frac{1}{2}v^2\right),$$

where r and r_w are dimensionless constants of order 1.

Setting these expressions equal and substituting [4] gives the final boundary condition:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}v^2\right) = \frac{r_w}{r}\frac{t}{t_w}\frac{1}{d}\left[v_w^2 + u_s^2\langle k_\tau^2\rangle - \frac{(1 - e_w)}{(1 + e_w)}v^2\right]\frac{v}{(v^2 + v_w^2 + u_s^2\langle k_\tau^2\rangle)^{1/2}}. \qquad [8]$$

STEADY-STATE COUETTE FLOW

In gravity-free, steady-state Couette flow, the walls are driven at a given velocity parallel to their surfaces. Figure 2 illustrates the geometry of the system; the plates are of infinite extent in the x- and z-directions with the origin of the y-axis midway between them. The number of grains in the channel will be specified by the parameter Δh, the free space remaining when all of the grains are packed towards one wall (figure 3). Due to symmetry the flow variables will be functions of y only. Combining this with the condition $\nabla \cdot \mathbf{u} = 0$ leads to the conclusion that the only nonzero component of the flow velocity $\mathbf{u}$ is u_x.

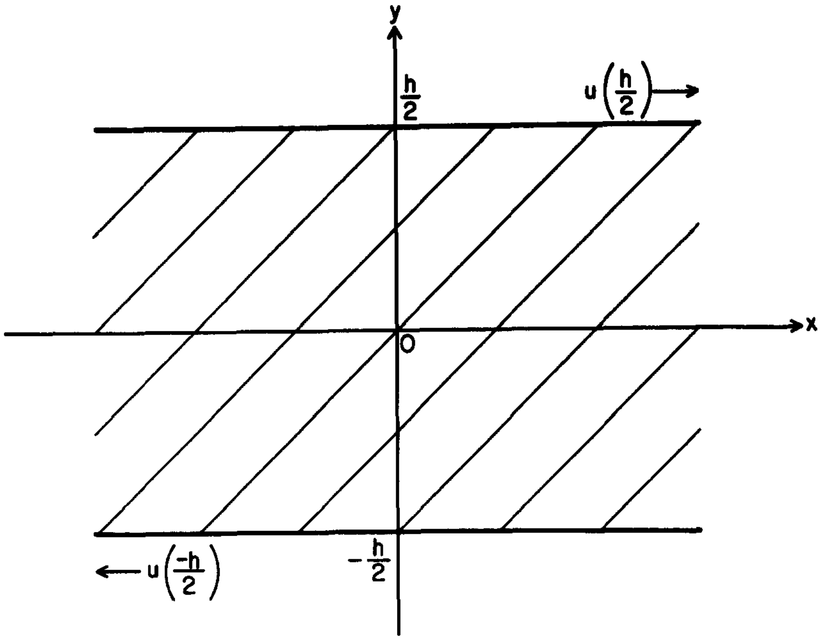

Figure 2. Illustration of the Couette flow geometry.

The equation governing the evolution of momentum in a steady granular flow is (Haff 1983):

$$\frac{\partial}{\partial t}(\rho u_i) = \frac{-\partial}{\partial x_k}\left[p\delta_{ik} + \rho u_i u_k - \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right].$$

Evaluating the x- and y-components of this equation gives the respective results that the shear stress $[\sigma_0 = \eta(\partial u_x/\partial y)]$ and the pressure (p_0) are constant throughout the flow.

The equation for the total kinetic and thermal energy of the system is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2\right) = \frac{-\partial}{\partial x_k}\left[\rho u_k\left(\frac{p}{\rho} + \frac{1}{2}u^2 + \frac{1}{2}v^2\right) - u_i\eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - K\frac{\partial}{\partial x_k}\left(\frac{1}{2}\rho v^2\right)\right] - I.$$

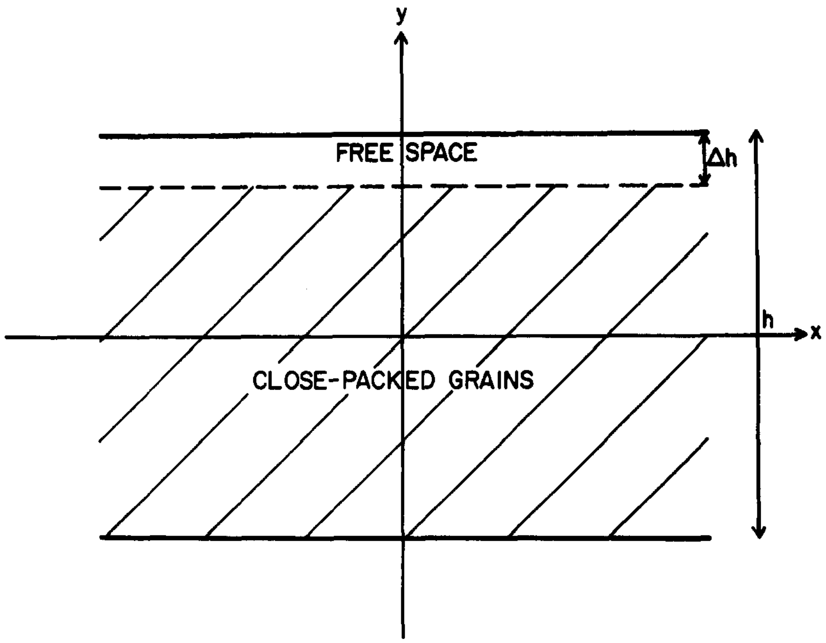

Figure 3. Illustration defining the free-space parameter Δh.

Setting to zero the derivatives with respect to t, x and z, and setting $u_y = u_z = 0$ leaves

$$0 = \frac{-\partial}{\partial y}\left[u_x \eta \frac{\partial u_x}{\partial y} - K\frac{\partial}{\partial y}\left(\frac{1}{2}\rho v^2\right)\right] - I.$$

The coefficient of thermal diffusivity (K) and the thermal energy sink (I) are given by

$$K = rd^2\frac{v}{s} \quad \text{and} \quad I = \gamma\rho\frac{v^3}{s},$$

where γ is a dimensionless constant related to the coefficient of restitution of a particle–particle collision e [$\gamma \propto (1 - e^2)$].

Substituting for these and for the constant shear stress and pressure gives the equation for the thermal velocity within the granular flow:

$$0 = \frac{\partial^2 v}{\partial y^2} + \omega^2 v, \qquad [9]$$

where

$$\omega^2 = \frac{1}{rd^2}\left(\frac{t^2\sigma_0^2}{qp_0^2} - \gamma\right). \qquad [10]$$

The general solution for the thermal velocity is

$$v(y) = 2v_0\cos(\omega y). \qquad [11]$$

Both v_0 and ω (i.e. σ_0/ρ_0) are determined by the boundary conditions. (Note that this assumes $\omega^2 > 0$; for the case of $\omega^2 < 0$, the trigonometric functions in this and succeeding equations are replaced by the corresponding hyperbolic functions.)

Combining [3] (with pressure equal to p_0) and [5] (with shear stress equal to σ_0), we can solve for the slip velocity in terms of the thermal velocity of the grains at the wall:

$$u_s = v(h/2)\frac{\sigma_0 t_w}{p_0 q_w\langle k_\tau^2\rangle}\left(1 - \frac{\sigma_0^2 t_w^2}{p_0^2 q_w^2\langle k_\tau^2\rangle}\right)^{-1/2}. \qquad [12]$$

Substituting this into [8] gives

$$\omega dD\left[\tan\left(\frac{\omega h}{2}\right)\right] = \left[E - \left[\frac{\dfrac{F\sigma_0^2}{p_0^2}}{1 - \dfrac{F\sigma_0^2}{p_0^2}}\right]\right]\left(1 - \frac{F\sigma_0^2}{p_0^2}\right)^{1/2}, \qquad [13]$$

where $D = (rt_w)/(r_w t)$, $E = (1 - e_w)/(1 + e_w)$ and $F = t_w^2/(q_w^2\langle k_\tau^2\rangle)$. This is a transcendental equation for σ_0^2/p_0^2 in terms of the particle properties (d, q, r, t and γ), the wall properties (e_w, q_w, r_w, t_w and $\langle k_\tau^2\rangle$) and the inter-wall spacing h. The equation can be solved numerically to obtain the shear stress to pressure ratio as a function of wall roughness $\langle k_\tau^2\rangle$. An example is given in figure 4. This ratio represents the tangent of the effective dynamical internal friction angle of the grain-mass-plus-wall system, which is essentially zero for nearly smooth walls, since the walls offer almost no resistance to shear. As the walls are roughened, the coupling between the walls and the grain mass increases [as evidenced by the rapidly decreasing slip velocity (figure 5)], finally reaching a point of saturation beyond which increasing wall roughness has little effect. This presumably reflects the fact that the main grain mass has become much weaker to shear than the wall–grain layer. This is a significant result because it means that if the walls of a shear cell apparatus designed to measure internal stresses are insufficiently rough, the measurements relate principally not to the effective internal friction in the bulk, but to the effective friction offered by the wall.

The average flow velocity is obtained from integrating $du/dy = \sigma_0/\eta$:

$$u(y) = \frac{2t}{q}\frac{\sigma_0}{p_0}\frac{v_0}{\omega d}\sin(\omega y), \qquad [14]$$

where now $u \equiv u_x$.

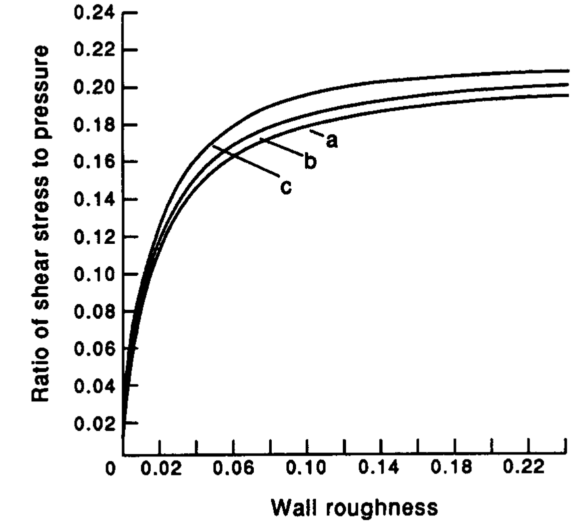

Figure 4. Ratio of shear stress to pressure, σ_0/p_0, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity. The values of the constants used in this numerical simulation are: $\gamma = 0.16$, $q = 0.25$, $r = 1.0$, $t = 1.0$, $e_w = 0.96$ (curve a), $e_w = 0.92$ (curve b), $e_w = 0.84$ (curve c), $q_w = 1.0$, $r_w = 1.0$ and $t_w = 0.5$. No stress is transmitted for perfectly smooth walls. As wall roughness increases, the system becomes more resistant to shearing until the shear resistance of the granular fluid itself becomes the determining factor.

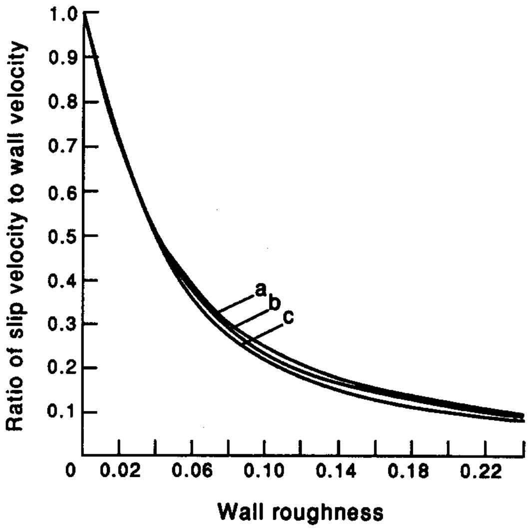

Figure 5. Ratio of slip velocity to wall velocity, u_s/u_w, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity. The slip velocity decreases with increasing roughness.

The wall velocity (with respect to the center of the channel where the flow velocity vanishes) is the sum of the slip velocity and the flow velocity at the wall:

$$u_w = 2v_0 \left\{ \cos\left(\frac{\omega h}{2}\right) \left[\frac{\dfrac{F\sigma_0^2}{p_0^2}}{\langle k_\tau^2 \rangle \left(1 - \dfrac{F\sigma_0^2}{p_0^2}\right)} \right]^{1/2} + \frac{t}{q} \frac{\sigma_0}{p_0} \frac{1}{\omega d} \sin\left(\frac{\omega h}{2}\right) \right\}. \quad [15]$$

The ratio of slip velocity to wall velocity is independent of v_0, and is shown in figure 5. The slip velocity decreases with increasing surface roughness as expected.

The ratio of adsorbed thermal energy flux to generated thermal energy flux at the wall is also independent of v_0, and is given by

$$\frac{Q_a}{Q_g} = \frac{E\left(1 - \dfrac{F\sigma_0^2}{p_0^2}\right)}{\dfrac{F\sigma_0^2}{p_0^2}}.$$

This ratio is plotted vs wall roughness in figure 6.

The value of v_0 is determined by the number of grains in the channel, as expressed by the parameter Δh. Beginning with the expression for Δh in terms of $s(y)$:

$$\Delta h = \frac{3}{d} \int_{-h/2}^{h/2} s(y)\, \mathrm{d}y;$$

substituting $s(y) = td\rho v^2(y)/p_0$ and integrating gives

$$v_0^2 = \frac{\omega\, \Delta h p_0}{6t\rho[\sin(\omega h) + \omega h]}. \quad [16]$$

Combining [11] and [15] gives the ratio of particle thermal velocity at the wall to wall velocity, which is shown in figure 7. At zero wall roughness no coupling of the wall to the fluid is possible, and the thermal velocity then vanishes. With increasing wall roughness more and more energy is transmitted to the grain mass and the thermal velocity near the wall increases. However, the slip

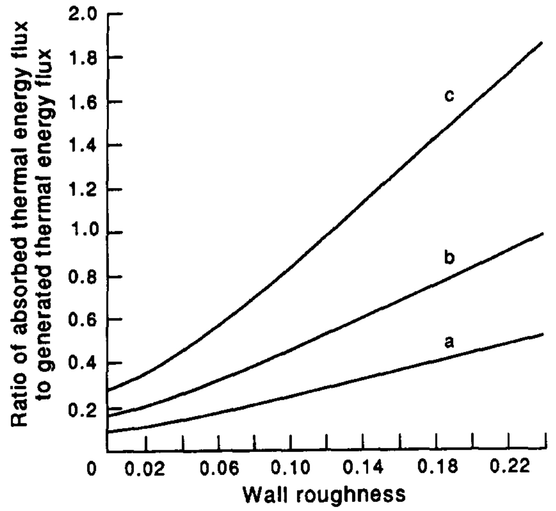

Figure 6. Ratio of absorbed thermal energy flux at the wall to the energy flux generated there, Q_a/Q_g, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity.

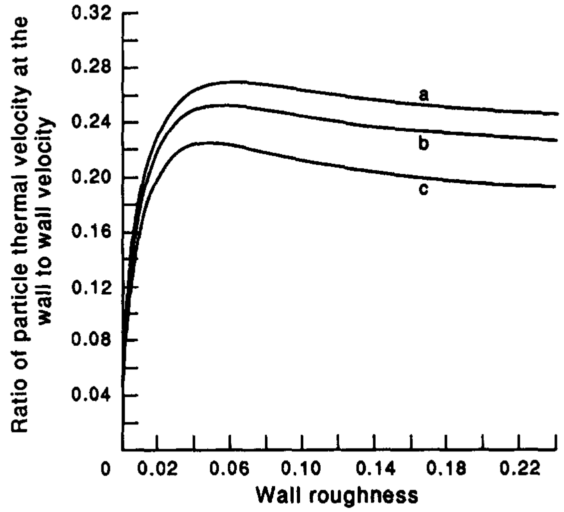

Figure 7. Ratio of particle thermal velocity at the wall to wall velocity, $v(h/2)/u_w$, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity. The thermal velocity is zero for perfectly smooth walls since in that case no energy can be transmitted from the wall to the fluid. After increasing with increasing roughness, v/u_w falls slightly (in this particular example) because the increase in stress does not quite compensate the decrease in slip velocity, and it is the product of stress and slip velocity which is ultimately responsible for generated heat at the wall.

velocity, which is the source of thermal energy, steadily decreases with increasing $\langle k_\tau^2 \rangle$ (figure 5), and the plot of thermal velocity vs wall roughness shows a maximum, with the thermal velocity slowly decreasing at high values of the roughness. [Whether a maximum always exists is not clear, since, while u_s decreases, σ is increasing with $\langle k_\tau^2 \rangle$ (figure 8), and it is their product which determines the energy generation rate.]

Using [15] and [16] we obtain the normalized shear stress

$$\frac{\sigma_0}{\rho_p d^2\left(\dfrac{2u_w}{h}\right)^2} = \frac{\dfrac{3}{8} t \dfrac{\sigma_0}{p_0} \dfrac{h^2}{d\,\Delta h}\left[\dfrac{\sin(\omega h)}{\omega d} + \dfrac{h}{d}\right]}{\left\{\cos\left(\dfrac{\omega h}{2}\right)\left[\dfrac{\dfrac{F\sigma_0^2}{p_0^2}}{\langle k_\tau^2 \rangle\left(1 - \dfrac{F\sigma_0^2}{p_0^2}\right)}\right]^{1/2} + \dfrac{t}{q}\dfrac{\sigma_0}{p_0}\dfrac{1}{\omega d}\sin\left(\dfrac{\omega h}{2}\right)\right\}^2},$$

where ρ_p is the particle material density. Setting h equal to 10 particle diameters and Δh equal to 3.25 particle diameters (corresponding to a volume fraction of 0.5) we obtain plots of the normalized shear stress and normalized pressure as functions of wall roughness (figures 8 and 9). Both shear stress and pressure vanish for perfectly smooth walls and increase with $\langle k_\tau^2 \rangle$ as expected.

The variations in flow properties across the channel are shown in figures 10(a–c), the character of the flow changing as the wall roughness varies. For small values of $\langle k_\tau^2 \rangle$ (nearly smooth walls) the shear stress to pressure ratio is small (figure 4) and $\omega^2 < 0$. In this case, the walls act as a net source of thermal energy flux ($Q_a/Q_g < 1$), and the thermal velocity drops as we move from the wall to the center of the flow [figure 10(a), curve 1]. The flow velocity has a (slight) inflection point [figure 10(b), curve 1] and the density is greatest in the center of the channel [figure 10(c), curve 1]. As the wall roughness is increased, we may reach the special solution called simple shear flow. Here the thermal velocity [figure 10(a), curve 2] and the particle–particle separation [figure 10(c), curve 2] are constant throughout the flow; the flow velocity increases linearly with distance [figure 10(b), curve 2]; and the walls act as neither a source nor a sink of thermal energy flux ($Q_a/Q_g = 1$). For

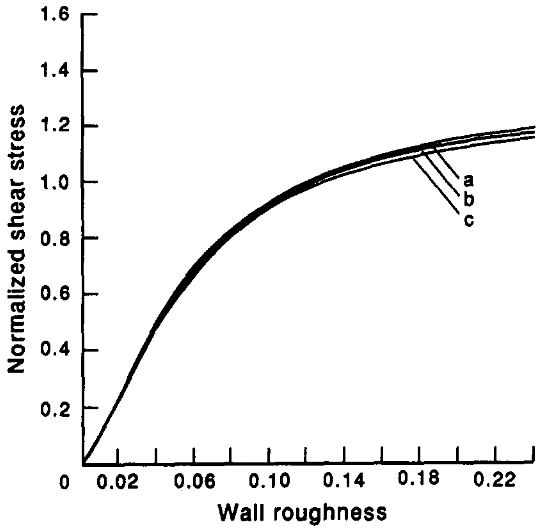

Figure 8. Normalized shear stress, $\{\sigma_0/[\rho_p d^2(2u_w/h)^2]\}$, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity with $h = 10$ grain diameters and $\Delta h = 3.25$ grain diameters. There can be no transmitted stress for perfectly smooth walls. Stress then increases as $\langle k_\tau^2 \rangle$ increases.

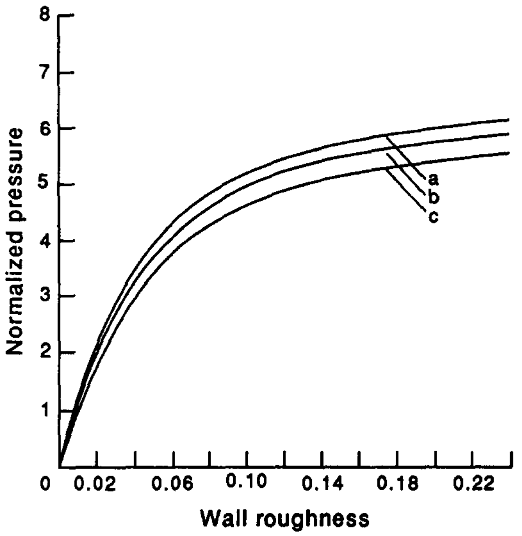

Figure 9. Normalized pressure, $\{p_0/[\rho_p d^2(2u_w/h)^2]\}$, vs wall roughness, $\langle k_\tau^2 \rangle$, at a fixed wall velocity. Like shear stress, the pressure must vanish for perfectly smooth walls and increase with increasing $\langle k_\tau^2 \rangle$.

granular systems, simple shear flow is not a common condition; it is only achieved by careful "tuning" of the wall parameters. Finally, for sufficiently rough walls, the shear stress to pressure ratio may be large enough that the rate of thermal energy generation in the flow exceeds the rate of thermal energy loss. This leads to a thermal velocity profile which has a maximum in the center of the flow [i.e. $\omega^2 > 0$; figure 10(a), curve 3], with the walls acting as a net energy sink ($Q_a/Q_g > 1$). In this case the flow velocity again shows a mid-channel inflection [figure 10(b), curve 3], while the density is a minimum in the center of the channel [figure 10(c), curve 3]. In general, the shape of the solution to the energy equation as a function of the strength of microscopic processes (r, q, t etc.) can be determined directly from [10] and [13].

Some effects on the thermal velocity of changes in the constants q (the dimensionless constant in the coefficient of viscosity), t_w (the dimensionless constant in the equation of state for particle–wall collisions) and e_w (the coefficient of restitution for particle–wall collisions) are shown in figures 11(a), (b) and (c), respectively. The particle constants and wall roughnesses are the same as in figures 10(a–c), except as noted. The curves in figures 11 (a–c) should be compared with those in figure 10(a). In figure 11(a), q has been increased by a factor of 2. An increase in q means that a lower shear rate can sustain a higher shear stress, leading to more slip at the wall. This increased slip results in a greater input of thermal energy from the wall to the flow, giving higher thermal velocities at the wall. In figure 11(b), t_w has been increased by a factor of 2. An increase in t_w means the pressure on the wall can be sustained with a larger particle–particle spacing at the wall, again leading to a higher slip velocity. The increase in the flow of thermal energy away from the wall can be seen. In figure 11(c), e_w has been increased from 0.84 to 0.96. An increase in e_w means that less energy is dissipated in particle–wall collisions. This will also result in a higher flux of thermal energy from the walls to the grains, as shown in the figure. In sum, modest changes of the system constants within this range of expected values lead to modest changes in the associated flow fields.

Finally, we note that throughout most of the range of $\langle k_\tau^2 \rangle$, the thermal velocity of the particles at the wall is a nearly constant fraction of the wall velocity. This constant fraction decreases as the walls are made more lossy (i.e. as e_w is reduced). The slip velocity as a fraction of wall velocity is seen to decrease with increasing wall roughness, as is expected. However, the lossiness of the walls has little effect on the slip velocity. Thus, it appears possible to set the slip velocity and the thermal velocity at the wall independently by adjusting the roughness and the lossiness of the wall.

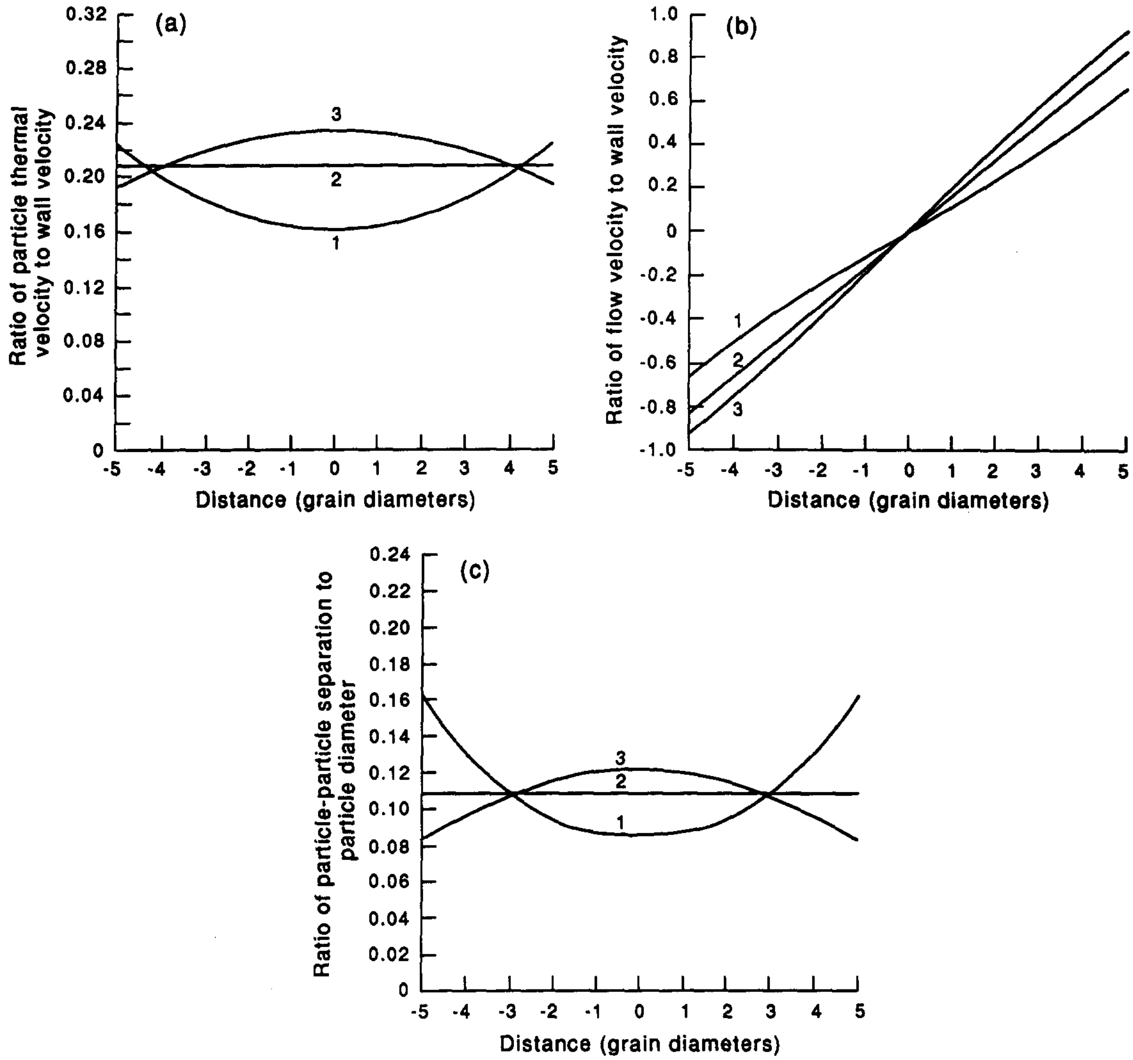

Figure 10. Variations in flow properties across the channel for several wall roughness values. The values of the constants are the same as in figure 4, curve c. The wall roughness values are $\langle k_\tau^2 \rangle = 0.0625$ (curve 1), $\langle k_\tau^2 \rangle = 0.125$ (curve 2) and $\langle k_\tau^2 \rangle = 0.25$ (curve 3); (a) the particle thermal velocity to wall velocity ratio v/u_w; (b) the flow velocity to wall velocity ratio u/u_w; (c) the particle–particle separation to particle diameter ratio s/d. See the text for a discussion.

CONCLUSION

We have presented a self-consistent set of boundary condition equations for a three-dimensional system of smooth, spherical particles slipping along a rough, possibly vibrating wall. These boundary conditions are based upon the postulated existence of slips in the bulk density, the average velocity and the thermal velocity at the wall. These slips represent first-order approximations to boundary effects in granular flows. We have solved the problem of Couette flow for the case of fixed wall velocity and particle number. In order to solve other problems of interest (such as flow down an inclined plane) the boundary conditions at a free surface will have to be developed.

One area for further study involves the effects of particle spin. Their inclusion will require the introduction of additional slips and boundary condition equations, as suggested by two-dimensional particle dynamics simulations (Campbell & Gong 1987). Also, we have assumed that walls with large roughness elements will tend to "heal" themselves through the trapping of particles. This trapping process has been discussed qualitatively (Hui *et al.* 1984; Haff 1987), but needs to be investigated in more detail.

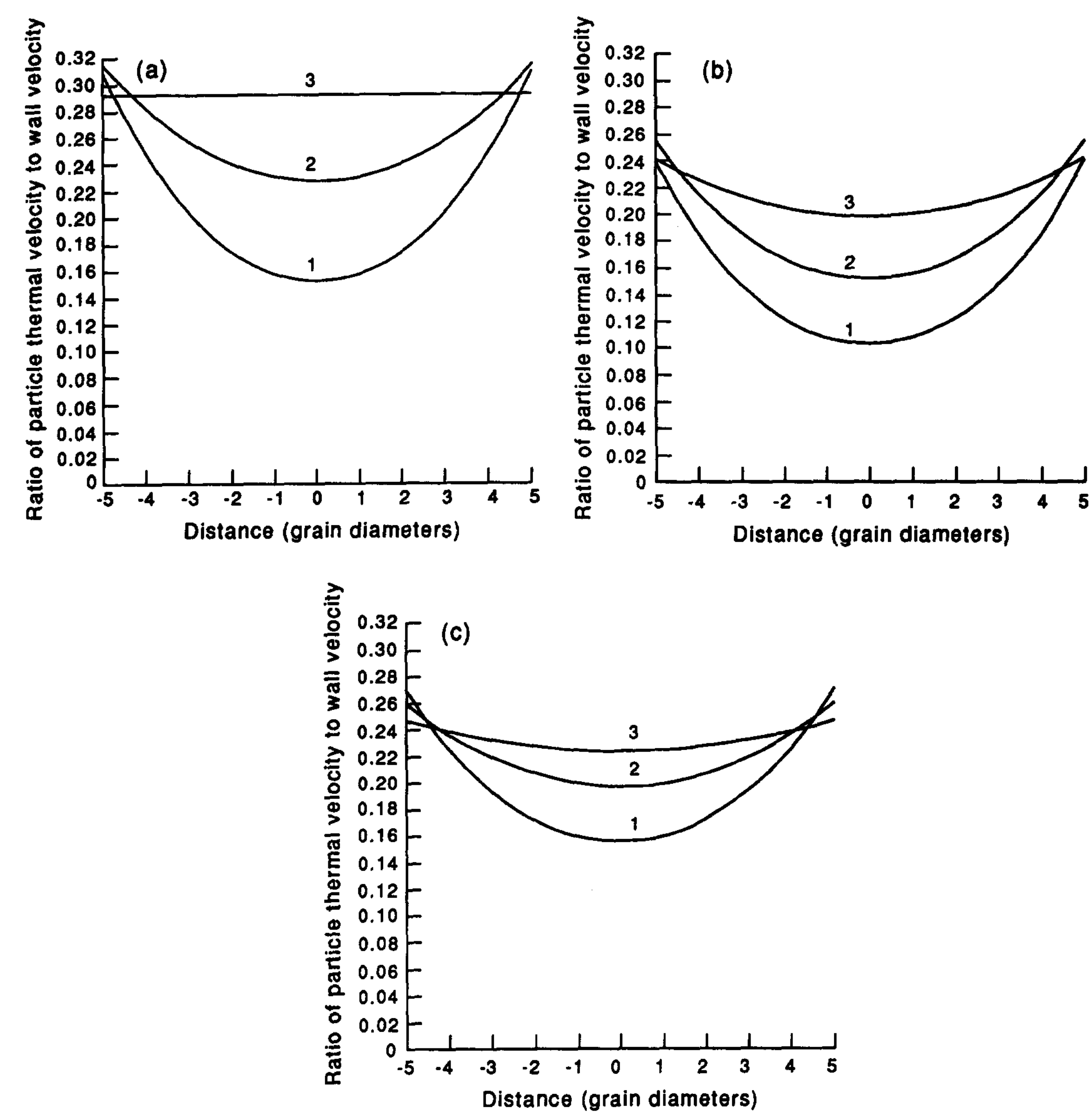

Figure 11. Variations in particle thermal velocity with changes in particle or wall properties. The values of the constants and wall roughness are the same as in figures 10(a–c), except: $q = 0.5$ in part (a), $t_w = 1.0$ in part (b) and $e_w = 0.96$ in part (c). See the text for a discussion.

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